

# Pooling and Insurance

## Actuarial Techniques - Pooling and Insurance<sup>1</sup>

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# Pooling and Insurance

- In the following sections we illustrate the effect of pooling, and then insurance, on the level of risk for an individual.
- We will follow those steps:
  - Define a situation, where risks are independent
  - Explore possible outcomes from simulation, comparing the cases without and with pooling
  - Analyse that situation mathematically
  - Extend this to a situation where risks are dependent
  - Contrast a pure pool with an insurance company, and explain how insurance adds to pooling
- The central question is: ***What is the basic idea behind insurance?***

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## 2 Risk pooling in case of independent risks

- Assumptions
- Example with simulations
- Example with formulas
- With pooling

# Independence

## Independence:

- Means information about one variable carries no information about the other ( $F(x, y) = F(x)F(y)$ )
- Is sometimes a reasonable assumption.
- For instance: the risk of dying for most individuals is an independent risk since the chance that a person dies does not usually depend on whether or not someone else has died
- There are exceptions, usually related to extreme events / disasters (such as a pandemic)

## Assumptions for the illustration

Risks:

- $n$  independent risks  $X_i$  for  $i = 1$  to  $n$  (e.g.,  $n$  individuals)
- Each risk - a loss of  $L$  with probability  $q$  or no loss with probability  $(1 - q)$  (e.g. life insurance)
- Each risk can incur only one loss

$$\begin{aligned} \Pr[X_i = L] &= q \\ \Pr[X_i = 0] &= \underline{\underline{(1 - q)}} \end{aligned}$$

(e.g. one can die only once)



## 2 Risk pooling in case of independent risks

- Assumptions
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One possible year with  $n = 100$ ,  $q = 0.01$ ,  $L = 10000$

```
set.seed(1)
claims1 <- 10000 * rbinom(100, 1, 0.01)
print(claims1, width = 55)
```

##	[1]	0	0	0	0	0	0	0
##	[9]	0	0	0	0	0	0	0
##	[17]	10000	0	0	0	0	0	0
##	[25]	0	0	0	0	0	0	0
##	[33]	0	0	0	0	0	0	0
##	[41]	0	0	0	0	0	0	0
##	[49]	0	0	0	0	0	0	0
##	[57]	0	0	0	0	0	0	0
##	[65]	0	0	0	0	0	0	0
##	[73]	0	0	0	0	0	0	0
##	[81]	0	0	0	0	0	0	0
##	[89]	0	0	0	0	0	0	0
##	[97]	0	0	0	0	0	0	0

$A = \text{average}$   
 $= 100$

## Another one

```
claims2 <- 10000 * rbinom(100, 1, 0.01)
print(claims2, width = 55)
```

The output shows a vector of 100 simulated claims. The values are mostly 0, with one value of 10000. The output is annotated with red handwritten notes and a red circle. The value 10000 is circled in black. A red arrow points from the word "x10" to the circled 10000. The number 200 is written in red at the top left, top right, and bottom left of the output. A large red oval encloses the entire output.

##	[1]	0	0	0	10000	0	0	0	0
##	[9]	0	0	0	0	0	0	0	0
##	[17]	0	0	0	0	0	0	0	0
##	[25]	0	0	0	0	0	0	0	0
##	[33]	0	0	0	0	0	0	0	0
##	[41]	0	0	0	0	0	0	0	0
##	[49]	0	0	0	0	0	0	0	0
##	[57]	0	0	0	0	0	0	0	0
##	[65]	0	0	0	0	0	0	0	0
##	[73]	0	0	0	0	0	0	0	0
##	[81]	0	0	0	0	0	0	0	0
##	[89]	0	0	0	0	0	0	0	0
##	[97]	0	0	0	0	0	0	0	0





## Another one

```
claims5 <- 10000 * rbinom(100, 1, 0.01)
print(claims5, width = 55)
```

```
## [1] 0 0 0 0 0 0 0 0
## [9] 0 0 10000 0 0 0 0 0
## [17] 0 0 0 0 0 0 0 0
## [25] 0 0 0 0 0 0 0 0
## [33] 0 0 0 0 0 0 0 0
## [41] 0 0 0 0 0 0 0 0
## [49] 0 0 0 0 0 0 0 0
## [57] 0 0 0 0 0 0 0 0
## [65] 0 0 0 0 0 0 0 0
## [73] 0 0 0 0 0 0 0 0
## [81] 0 0 0 0 0 0 0 0
## [89] 0 0 0 0 0 0 0 0
## [97] 0 0 0 0 0 0 0 0
```

0000

$$\frac{10000}{100} = 100$$

## 2 Risk pooling in case of independent risks

- Assumptions
- Example with simulations
- **Example with formulas**
- With pooling

## In the example above

An individual has a 0.01 chance of dying and the insurer will pay \$10,000 upon their death. Calculate the expected death benefit and variance of the death benefit for this individual.

- Expected benefit

$$E(X) = 0.01 \times 10,000 = \underline{100}$$

*Handwritten notes: A blue circle around '100', a blue circle around '10000', and a blue arrow pointing from the '10000' circle to the '100' circle.*

- Variance

$$0.01 \times 0.99 \times 10,000^2 = 990,000 = 995^2$$

*Handwritten notes: A blue circle around '990,000', and a blue circle around '995'.*

- Standard deviation is  $\sqrt{990,000} = 995$ .
- Handwritten notes: A blue circle around '995', and a blue arrow pointing from the '995' circle to the '995' in the previous equation.*

This corresponds to the moments of this unique risk. Of course, insurers have more than one risk in their portfolio.

This leads to **pooling**.



## 2 Risk pooling in case of independent risks

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## Risk Pooling - How does it work?

- Pool the  $n$  risks
- Each individual agrees to pay the **actual** average of the total losses into the pool
- The pool pays the losses of the members of the pool who incur a loss
- Average of the losses

$$A = \frac{\sum_{i=1}^n X_i}{n} \quad \frac{10000}{100}$$

with  $X_i = L$  if a loss occurs or  $X_i = 0$  if no loss occurs.

**This is a random variable** - we do not know what it will be until the losses occur - The good luck of some compensates for the bad luck of others.

## Simulated example

In this case, the hundred individuals all pay the same amount, corresponding to the average. For the 5 cases, this is

```
print(c(mean(claims1), mean(claims2), mean(claims3), mean(claims4),
        mean(claims5)))
```

```
## [1] 100 200 0 200 100
```

A

- ① still random!
- ② less random!

# Expectation with pooling

Expected value of the average loss for the pool

$$E[A] = \mu_A = E\left[\frac{\sum_{i=1}^n X_i}{n}\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{n\mu}{n} = \mu = E[X]$$

Handwritten notes:  $E / \text{Var}$  with / without

Number of losses (in this case) has a Binomial  $(n, q)$  distribution, so that probability that  $j$  individuals incur a loss is

$$\Pr\left(\sum_{i=1}^n X_i = jL\right) = \binom{n}{j} q^j (1-q)^{n-j} \quad j = 0, 1, 2, \dots, n.$$

Handwritten notes:  $\Pr[N=j]$

Expected value of the average loss

$$\begin{aligned}
 \mu_A &= E\left[\frac{\sum_{i=1}^n X_i}{n}\right] = E\left[\frac{N \cdot L}{n}\right] \\
 &= \sum_{j=0}^n \frac{jL}{n} \binom{n}{j} q^j (1-q)^{n-j} \\
 &= \frac{L}{n} \sum_{j=0}^n j \binom{n}{j} q^j (1-q)^{n-j} \\
 &= \frac{L}{n} nq = qL \\
 &= \mu_i \\
 &= E[X_i]
 \end{aligned}$$

$E[N] = \sum_{j=0}^n j \cdot \Pr[N=j]$   
 $E[N] = n \cdot q$

Expected value of the average loss in the pool is the same as the expected value of the individual loss!

## Note - Expectation is a linear operator

We have

$$E[k] = k$$

where  $k$  is a constant, and

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

where  $X$  and  $Y$  are random variables that need not be independent, and  $a$ ,  $b$  and  $c$  are constants

## Revision - Properties of Variance

For any random variable  $X$

$$\text{Var}[X] = E[X^2] - E[X]^2$$

so that

$$E[X^2] = \text{Var}[X] + E[X]^2$$

and also

$$\text{Var}[kX] = k^2 \text{Var}[X]$$

Also,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

if  $X$  and  $Y$  are independent.

# Variance with pooling

$$\begin{aligned}
 \sigma_A^2 &= \text{Var} \left[ \frac{\sum_{i=1}^n X_i}{n} \right] \\
 &= \frac{1}{n^2} \text{Var} \left[ \sum_{i=1}^n X_i \right] \\
 &= \frac{1}{n^2} \sum_{i=1}^n \text{Var} [X_i] \quad \text{because the } X_i\text{'s are independent} \\
 &= \frac{n}{n^2} \text{Var} [X_i] \quad \text{because the } X_i\text{'s have same distribution} \\
 &= \frac{n}{n^2} L^2 \text{Var} [Y_i] \quad \text{where } Y_i \text{ is Bernoulli}(q) \\
 &= \frac{L^2}{n} q(1-q)
 \end{aligned}$$

$\text{Bin}(1, q)$   
 $\downarrow$   
 $\text{Var}(X_i) = \text{Var}(L \cdot N) = L^2 q(1-q)$   
 $N \sim \text{Bin}(n, q)$   
 $\text{Var}(N) = n \cdot q \cdot (1-q)$




## Risk Pooling - Pool Variance in Words

The variance of the average of the pool losses is the variance of each individual loss  $q(1 - q)L^2$  divided by the number of individuals in the pool

$$\sigma_A^2 = \frac{q(1 - q)L^2}{n} = \frac{\text{Var}(X)}{n}$$

## Risk Pooling - So Why Pool?

- For risk averse individuals, the “happiness” of each individual who participates in the risk pooling is higher (variability is lower)
- A mathematical model for explaining this, in microeconomics, would use so-called “utility functions”
- If they fully insure, they  exchange all their risk, otherwise they exchange only a proportion of it
- If the risk bearer had a choice of how much risk to transfer (e.g. an insurer vs an reinsurer, choice of excess on an insurance contract), this optimal proportion would depend on the cost of transfer and their risk aversion.

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### 3 Risk pooling in case of correlated risks

- Assumptions
- Pooling of Correlated Risks
- Effect of pooling
- Diversification benefits

# Assumptions

- Here we relax the assumption of independence, as this is a major assumption
- We consider the simplest dependence model possible: **correlation**
- The basic formulas about correlation are

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y),$$

and then the **coefficient of correlation** is defined as

$$-1 \leq \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\underbrace{\sqrt{\text{Var}(X)}}_{\sigma(X)} \underbrace{\sqrt{\text{Var}(Y)}}_{\sigma(Y)}} \leq 1.$$

### 3 Risk pooling in case of correlated risks

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# Pooling of Correlated Risks

ind  $\Rightarrow \rho = 0$   
 $-1 \leq \rho \leq 1$

Simplest case

- Two losses  $X$  and  $Y$  - each with the same expected value  $\mu$  and the same variance  $\sigma^2$
- Correlation  $\rho(X, Y) = \rho$ , equal variances with  $\sigma_X = \sigma_Y = \sigma$ . Hence  $\text{Cov}(X, Y) = \rho\sigma^2$ .

Average expected loss is

$$E\left[\frac{X+Y}{2}\right] = \frac{E[X+Y]}{2} = \frac{E[X] + E[Y]}{2} = \frac{2\mu}{2} = \mu$$

which is what we expected.

### 3 Risk pooling in case of correlated risks

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# Effect of pooling

Variance of the average loss is

$$\text{Var} \left[ \frac{X + Y}{2} \right] = \frac{1}{4} \left[ \sigma^2 + \sigma^2 + 2\rho\sigma^2 \right]$$

$$= \frac{\sigma^2}{2} (1 + \rho)$$

*independence*

*correlation*

$$(2 + 2\rho)\sigma^2$$

- Pooling of correlated risks multiplies the variance, relative to the variance if the risks were independent, by the factor  $(1 + \rho)$  where  $-1 \leq \rho \leq 1$ .
- If risks are perfectly correlated ( $\rho = 1$ ), the variance is

$$\text{Var} \left[ \frac{X + Y}{2} \right] = \underline{\sigma^2} \quad \frac{\sigma^2}{2} \cdot 2 = \underline{\underline{\sigma^2}}$$

Then there is no apparent benefit in pooling.

### 3 Risk pooling in case of correlated risks

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$\sigma < 1$ ?

## Diversification benefits

As soon as dependence is not perfect, there are **diversification benefits**:

$$\begin{aligned} \text{diversification benefit} &= \sigma^2 - \frac{\sigma^2}{2} (1 + \rho) \\ &= \sigma^2 \frac{1 - \rho}{2} \end{aligned}$$

The more the diversification benefits, the lower the risk, and the lower the capital requirements. Accurate calculation of diversification benefits is difficult but crucial.

$$\begin{aligned} \rho = 1 &\Rightarrow DB = 0 \\ \rho = 0 &\Rightarrow DB = \frac{\sigma^2}{2} \\ \rho = -1 &\Rightarrow DB = \sigma^2 \end{aligned}$$

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# References I

Atkinson, M. E., and David C. M. Dickson. 2011. *An Introduction to Actuarial Studies*. 2nd ed. Edward Elgar.