Compound Interest - Annuities and The Equation of Value

Actuarial Techniques - Time Value of Money[1]

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[1] References: Section 2.2.2-3 of Atkinson and Dickson (2011) | → latest slides

Valuing annuities

Finding the present value of a series of equal payments = **"annuities"**:

- So far we have considered the calculation of the present value and the accumulated value of single cash flows.
- It is quite common in practice for the same cash flows to be repeated many times. For example,
  - a new home owner may repay the bank $500 every fortnight over a 25 year period
  - or a life insurance company might pay a retiree $1,500 every month for the rest of the person's life.
- In order to value (that is, to find the present value of) a series of payments, we could find the present value of each individual payment in the series of payments and sum the resulting series.
- This approach will very quickly become tedious for long series of cash flows. We therefore develop formulae for finding the present value of streams of equal payments.

2. Annuity in arrears – Annuity-Immediate
   - Definition
   - Example 1

Definition

Consider the figure below which contains n years. Consider the case where a payment of \$1 is made at the end of each of the n years marked in the diagram. Suppose we want to find the present value of these payments – that is, their value at time 0.

We perform this calculation very often and so we give a symbol for the present value result. Define

$$a_{\overline{n}|} = v + v^2 + \ldots + v^n \tag{1}$$

This type of annuity is called an **annuity in arrears** because the payments are made at the end of each year.

2 Annuity in arrears – Annuity-Immediate

- Definition
- Example 1

Example 1

1. Write down an expression in terms v for $(1+i)a_{\overline{n}|}$.
2. Use part (1) and the definition of $a_{\overline{n}|}$ to derive an expression for $ia_{\overline{n}|}$. Hence show that

$$a_{\overline{n}|} = \frac{1-v^n}{i}. \tag{2}$$

3. If we rearrange our result from (2), we get

$$1 = ia_{\overline{n}|} + v^n.$$

Interpret this expression in terms of a loan of \$1 and the associated repayments.

Solution to Part 1

We have

$$(1+i)a_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1}$$

Solution to Part 2

Subtract the definition from the result in (1) to get

$$ia_{\overline{n}|} = 1 - v^n$$

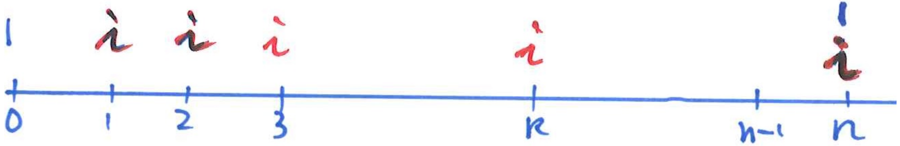
Make $a_{\overline{n}|}$ the subject and the result is proved.

Solution to Part 3

We need to interpret

$$1 = ia_{\overline{n}|} + v^n$$

For a loan of $1 today, we can repay interest (only) on the loan at the end of each year for n years and then repay the amount borrowed ($1) in n years' time.



The present value of the borrowings (\$1) must equal the present value of the interest repayments ($i\, a_{\overline{n}|}$) plus the present value of the repayment of principal, v^n, i.e.

$$1 = i a_{\overline{n}|} + v^n \quad \text{and hence} \quad a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

Part 3: A concrete example

- Take $n = 10, i = 5\%$. That is to say,

$$100 = 5a_{\overline{10}|} + 100v^{10}$$

- Suppose I lend you \$100 and charge you interest at 5% per annum effective. This means your debt in one year's time will be \$105.
- Suppose you repay \$5 at the end of 1 year – you have repaid the interest only, and still owe me \$100.
- At the end of the second year you again repay \$5 – your debt is still \$100.
- Suppose this pattern continues for 10 years – each year you pay \$5 at the year end, and at the end of the 10th year you repay the \$100.
- Your repayments can be represented as a 10 year annuity in arrears of amount 5, and a payment of 100 at time 10 years.
- As my loan has been repaid, the present value of my outgo equals that of my income, i.e.

$$100 = 5a_{\overline{10}|} + 100v^{10}.$$

Part 2: Alternative derivation

How else could you derive the result in (2)?

By using the sum of a geometric progression with n terms, first term v and common ratio equal to v, we get

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + \ldots + v^n \\ &= v(1 + v^2 + \ldots + v^{n-1}) \\ &= v\frac{1-v^n}{1-v} \\ &= \frac{1-v^n}{(1/v)-1}. \end{aligned}$$

As

$$1/v = 1 + i,$$

we get

$$a_{\overline{n}|} = \frac{1-v^n}{i}.$$

Definition

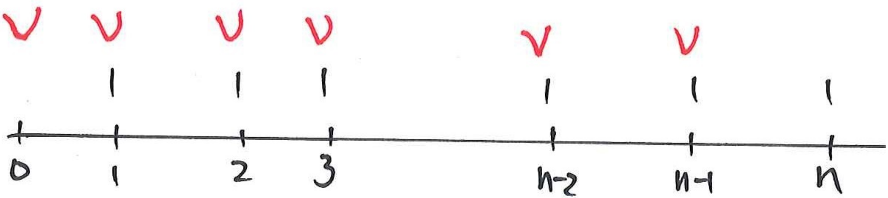
- The annuity we considered above is called an annuity in arrears since the payments are made at the end of each of the n years.
- Suppose instead that the payments are made at the start of each of the n years.
- The annuity is then called an *annuity in advance*, or *annuity-due*.
- Mark the payments and the valuation date on the diagram below for an n year annuity in advance.
- The present value of an annuity in advance is written as $\ddot{a}_{\overline{n}|}$ and is pronounced "a due n."

3 Annuity in advance – Annuity-Due

- Definition
- Example 2
- Example 3

Example 2

1. Write down a sum in terms of v for $\ddot{a}_{\overline{n}|}$.
2. Explain in words why
$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}.$$
3. Show that
$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}.$$



Solution to Part 1

We have

$$\ddot{a}_{\overline{n|}} = 1 + v + v^2 + \ldots + v^{n-1}.$$

Solution to Part 2

The number of payments and the valuation interest rates are the same. The only difference is that payments occur one year earlier in $\ddot{a}_{\overline{n}|}$ as compared with $a_{\overline{n}|}$, which explains why it is worth more, by a factor or $(1+i)$.

Solution to Part 3

Using our definition for $a_{\overline{n}|}$, we can write

$$a_{\overline{n-1}|} = v + v^2 + ... + v^{n-1}.$$

This clearly includes every term in the summation for $\ddot{a}_{\overline{n}|}$ except the initial payment of 1. Therefore

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}.$$

3 Annuity in advance – Annuity-Due

- Definition
- Example 2
- **Example 3**

Example 3

1. Find the PV of 20 annual payments of \$1,000 at 6% per annum effective with the first payment due in 12 months' time.
2. Find the PV of 15 annual payments of \$700 at 5% per annum effective with the first payment due immediately.

Solutions:

1. $$1,000a_{\overline{20}|} = 1,000\frac{1-1.06^{-20}}{0.06} = 11,469.92.$$

2. $$700\ddot{a}_{\overline{15}|} = 700(1.05)a_{\overline{15}|} = 7,629.05.$$

Some reasonableness checks for annuities

Quite often in actuarial work, we are involved with very complex calculations. It is useful to be able to place a rough check (a reasonableness check) on our work at the end. This can be a useful way to remove any careless errors that we may have been made during the course of our work.

4 Some reasonableness checks for annuities

- First check
- Second check
- Third check
- Example 4

First check

First, (assuming $i \geq 0$)

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + v^3 + \ldots + v^n \\ &\leq 1 + 1 + 1 + \ldots + 1 = n. \end{aligned}$$

Thus, **the value of $a_{\overline{n}|}$ cannot exceed** n.

4 Some reasonableness checks for annuities

- First check
- **Second check**
- Third check
- Example 4

Second check

Second, (assuming $i > 0$)

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \to \frac{1}{i} \text{ as } n \to \infty$$

Thus, **the value of $a_{\overline{n}|}$ cannot exceed** $1/i$.

Third check

Third, consider the amounts and the timing of the payments:

- Each of the n payments is of amount 1, so the total amount is n.
- The payment times are $1, 2, \ldots, n$, so the average payment time is

$$\frac{1}{n}(1+2+\ldots+n) = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

  Hence, **an approximation of $a_{\overline{n}|}$ is**

$$a_{\overline{n}|} \approx nv^{\frac{n+1}{2}}.$$

For example, compare at 6%:

$$a_{\overline{20}|} = \frac{1-1.06^{-20}}{0.06} = 11.4699,$$

while

$$20 \times 1.06^{-10.5} = 10.85.$$

4 Some reasonableness checks for annuities

- First check
- Second check
- Third check
- Example 4

Example 4

1. Calculate the PV of a series of payments of \$100 at the end of each of the next 20 years at 9% per annum effective.
2. Calculate the PV of a series of payments of \$100 at the beginning of each of the next 15 years at 8% per annum convertible quarterly.
3. Apply a reasonableness check to your answer to Part (2).

Solutions:

1. The PV is
$$100a_{\overline{20}|} = 100\frac{1-1.09^{-20}}{0.09} = 912.85$$
2. The PV is
$$100\ddot{a}_{\overline{15}|} = 100(1+i)\frac{1-(1+i)^{-15}}{i} = 912.90,$$
where $1+i = 1.02^4$.
3. The PV is approximately
$$100 \times 15v^7 = 1,500 \times 1.02^{-28} = \$861.$$
$(\frac{1}{15} \times \frac{1}{2} \times 14 \times 15 = 7)$

5 Perpetuities

- Definition
- Example 5

Definition

- In fact, this infinite term annuity seen before is called a *perpetuity*, and is written

$$
\begin{aligned}
a_{\overline{\infty}|i} &= v + v^2 + v^3 + \ldots \\
&= v(1 + v + v^2 + v^3 + \ldots) \\
&= \frac{v}{1-v} \\
&= \frac{1}{i}
\end{aligned}
$$

- This can also be used as a building block to derive annuity formulas.

5 Perpetuities

- Definition
- Example 5

Example 5

Derive the formula for an annuity-immediate

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

using perpetuity

$$a_{\overline{\infty}|i} = \frac{1}{i}.$$

Solution:

- Consider the difference between two infinite series of payment of 1, one starting now with present value

$$a_{\overline{\infty}|i} = \frac{1}{i},$$

and one starting in n years with present value

$$v^n a_{\overline{\infty}|i} = v^n \frac{1}{i}.$$

- The difference in cash flows corresponds exactly to that of an annuity-immediate over n years, with present value

$$\frac{1}{i} - v^n \frac{1}{i} = \frac{1 - v^n}{i} = a_{\overline{n}|},$$

as required.

6 Deferred annuities

- Definition
- Example 6
- Formula ${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$
- Example 7

Definition

We now develop formulae for annuities where the first payment is delayed by m years. We again consider the case where n payments are made in total.

Consider first the annuity in arrears. In the non-deferred case, the first payment is made at the end of the first year, that is, at time 1. In the deferred annuity, the first payment is therefore made at time $m+1$, that is, at the end of the $(m+1)$st year.

$a_{\overline{n}|} = v + v^2 + \cdots v^n$ $\qquad$ ${}_{m|}a_{\overline{n}|} = v^{m+1} + v^{m+2} + \cdots v^{m+n}$

The notation used for the present value of an m year deferred, n year annuity with payments in arrears is ${}_{m|}a_{\overline{n}|}$.

6 Deferred annuities

- Definition
- Example 6
- Formula ${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$
- Example 7

Example 6

1. Write down an expression for ${}_{m|}a_{\overline{n}|}$ in terms of v.
2. Write an expression for ${}_{m|}a_{\overline{n}|}$ in terms of v, m and $a_{\overline{n}|}$.

Solutions:

1. $$ {}_{m|}a_{\overline{n}|} = v^{m+1} + v^{m+2} + \ldots + v^{m+n} $$
2. From above, the expression for ${}_{m|}a_{\overline{n}|}$ involves m years further discounting for each term than is required under $a_{\overline{n}|}$. Therefore, we have

$$ {}_{m|}a_{\overline{n}|} = v^m(v + v^2 + \ldots + v^n) = v^m a_{\overline{n}|}. $$

6 Deferred annuities

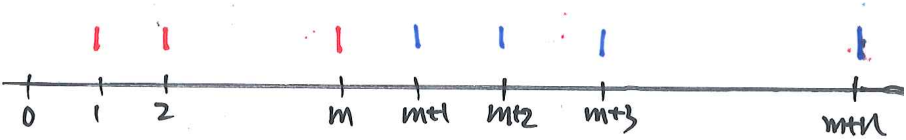
Proof of ${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$

Using the result in part (2) of Example 6 above, we have

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^m a_{\overline{n}|} = v^m \frac{1-v^n}{i} = \frac{v^m - v^{m+n}}{i} \\ &= \frac{1 - v^{m+n} - (1 - v^m)}{i} = a_{\overline{m+n}|} - a_{\overline{m}|}. \end{aligned}$$

Interpretation

$$ {}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} \iff a_{\overline{m+n}|} = a_{\overline{m}|} + {}_{m|}a_{\overline{n}|} $$



6 Deferred annuities

- Definition
- Example 6
- Formula ${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$
- Example 7

Example 7

Find the PV of a series of 15 payments of \$1 at yearly intervals beginning 11 years from now. Use an interest rate of 7% per annum effective.

- *Method One:* The PV is

$$ {}_{10|}a_{\overline{15}|} = v^{10}\,\frac{1-v^{15}}{i} = 1.07^{-10}\,\frac{1-1.07^{-15}}{0.07} = 4.63. $$

- *Method Two:* The PV is

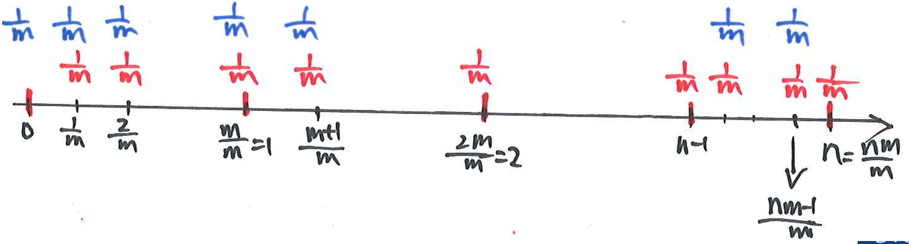
$$ a_{\overline{25}|} - a_{\overline{10}|} = \frac{1-1.07^{-25}}{0.07} - \frac{1-1.07^{-10}}{0.07} $$
$$ = 4.63. $$

- *Rough Check:* PV $\approx 15v^{18} = 4.44$.

7. Annuities payable in m partial payments
   - Annuities payable m-thly
   - m-thly annuity in arrears / annuity-immediate
   - m-thly annuity-due
   - Examples
   - Alternative derivations

Annuities payable *m*-thly

- We have already discussed annuities where payments are made for n years with one payment of \$1 each year.
- We now break up that single payment of \$1 into m payments each of \$ $1/m$
- Consider the following timeline and mark on the payments for a n year annuity with payments made m times per year, where each payment is made at the end of m-th of a year.



7 Annuities payable in m partial payments

- Annuities payable m-thly
- m-thly annuity in arrears / annuity-immediate
- m-thly annuity-due
- Examples
- Alternative derivations

m-thly annuity in arrears / annuity-immediate

The notation used for an m-thly annuity with payments at the end of each m-th of a year is

$$a^{(m)}_{\overline{n}|}.$$

The derivation of a formula for this annuity is as follows:

$$a^{(m)}_{\overline{n}|} = \frac{1}{m}\left(v^{1/m} + v^{2/m} + \ldots + v^{nm/m}\right)$$

and

$$(1+i)^{1/m} a^{(m)}_{\overline{n}|} = \frac{1}{m}\left(1 + v^{1/m} + \ldots + v^{(nm-1)/m}\right)$$

Subtracting the first identity from the second gives

$$a_{\overline{n}|}^{(m)}\left((1+i)^{1/m}-1\right)=\frac{1}{m}\left(1-v^{nm/m}\right)$$

or, equivalently,

$$a_{\overline{n}|}^{(m)}=\frac{1-v^{n}}{m\left((1+i)^{1/m}-1\right)}=\frac{1-v^{n}}{i^{(m)}}$$

7 Annuities payable in *m* partial payments

- Annuities payable *m*-thly
- *m*-thly annuity in arrears / annuity-immediate
- *m*-thly annuity-due
- Examples
- Alternative derivations

m-thly annuity-due

The notation used for an m-thly annuity with payments at the beginning of each m-th of a year is

$$\ddot{a}_{\overline{n}|}^{(m)}.$$

The derivation of a formula for this annuity is as follows:

$$\begin{aligned}
\ddot{a}_{\overline{n}|}^{(m)} &= \frac{1}{m}\left(1 + v^{1/m} + v^{2/m} + \ldots + v^{(nm-1)/m}\right) \\
&= v^{-1/m}\frac{1}{m}\left(v^{1/m} + v^{2/m} + \ldots + v^{nm/m}\right) \\
&= (1+i)^{1/m} a_{\overline{n}|}^{(m)}
\end{aligned}$$

7 Annuities payable in m partial payments

- Annuities payable m-thly
- m-thly annuity in arrears / annuity-immediate
- m-thly annuity-due
- Examples
- Alternative derivations

Example 8

Find the PV of $100 per annum payable quarterly in arrears for 10 years at 7.5% per annum effective.

Solution:

We have

$$100a_{\overline{10}|}^{(4)} = 100\frac{1 - 1.075^{-10}}{i^{(4)}} = 705.42,$$

where

$$i^{(4)} = 0.072978.$$

Example 9

Find the PV of \$100 per annum payable quarterly in arrears for 10 years given interest is 8% per annum convertible quarterly.

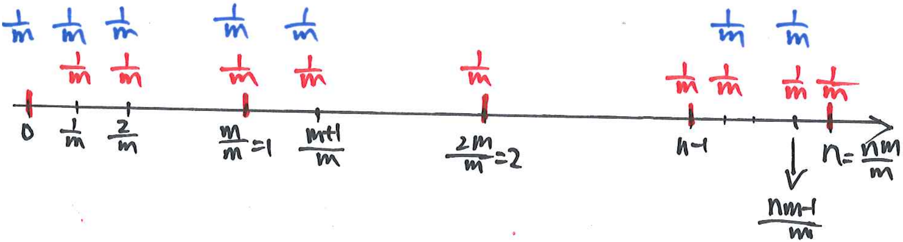
Solution:

$$\begin{aligned} 1+i &= 1.02^4 \\ 100a_{\overline{10}|}^{(4)} &= 100\frac{1-v^{10}}{i^{(4)}} \\ &= 100\frac{1-1.02^{-40}}{0.08} \\ &= 683.89 \end{aligned}$$

7 Annuities payable in m partial payments

- Annuities payable m-thly
- m-thly annuity in arrears / annuity-immediate
- m-thly annuity-due
- Examples
- Alternative derivations

An alternative expression for $a_{\overline{n}|}^{(m)}$



Changing the time unit,

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} = \frac{1}{m}\frac{1-(v^{1/m})^{nm}}{\frac{i^{(m)}}{m}}$$

$$= \frac{1}{m} a_{\overline{nm}|\, j}$$

In the above formula,

- The time unit is $1/m$ of a year;
- $j = i^{(m)}/m$ is the effective interest rate per time unit;
- $v^{1/m} = 1/(1+j)$ is the discount per time unit ($1/m$ of a year);
- $1/m$ is the amount of payment per unit of time;
- nm is the number of payments.

An alternative expression for $a_{\overline{n}|}^{(m)}$ – Using perpetuities

By analogy, we have

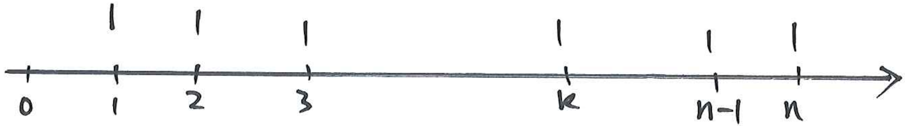
$$a_{\overline{\infty}|i^{(m)}} = \frac{1}{i^{(m)}}.$$

The result immediately follows.

8 Accumulated values of annuities

Definition

- Consider a superannuation fund into which payments are made during the working life of an individual. These payments will form a regular stream of payments and we are interested in knowing how much these payments will be worth at retirement, when interest is applied to each of the payments.
- We therefore are often interested in calculating the accumulated value of an annuity.
- $s_{\overline{n|}}$ represents the accumulation at time n of a series of payments of 1 at unit intervals in arrears (i.e. at times $1, 2, \ldots, n$). Mark the payments and the valuation date on the timeline below.



8 Accumulated values of annuities

- Definition
- Accumulated value of an annuity in arrears
- Relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$
- Accumulated values of an annuity-due
- Examples

Accumulated value of an annuity in arrears

We derive an expression for $s_{\overline{n}|}$ in a very similar way to how we derived formulae for present values of annuities:

$$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i) + 1$$

gives

$$(1+i)s_{\overline{n}|} = (1+i)^{n} + (1+i)^{n-1} + \ldots + (1+i),$$

and so

$$i s_{\overline{n}|} = (1+i)^{n} - 1.$$

Thus,

$$s_{\overline{n}|} = \frac{(1+i)^{n} - 1}{i}.$$

8 Accumulated values of annuities

- Definition
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Relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$

We now derive a relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$:

Result:

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}.$$

Proof:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} = (1+i)^n \frac{1 - v^n}{i} = (1+i)^n a_{\overline{n}|}.$$

8 Accumulated values of annuities

- Definition
- Accumulated value of an annuity in arrears
- Relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$
- **Accumulated values of an annuity-due**
- Examples

Accumulated values of an annuity-due

Suppose now that the n payments considered above are made at the start of each of the n years of the annuity. The payments are therefore made in advance.

We aim to find the value of these payments at time n. Mark the payments and valuation date on the diagram below.

Notation: $\ddot{s}_{\overline{n}|}$ represents the accumulation at time n of a series of payments of 1 at unit intervals in advance (i.e. at times $0, 1, 2, \ldots, n-1$).

8 Accumulated values of annuities

Example 10

1. Write down an expression in terms of i and n for $\ddot{s}_{\overline{n}|}$.
2. Show that

$$\ddot{s}_{\overline{n}|} = (1+i)^{n+1}\ a_{\overline{n}|} = (1+i)^n\ \ddot{a}_{\overline{n}|}$$

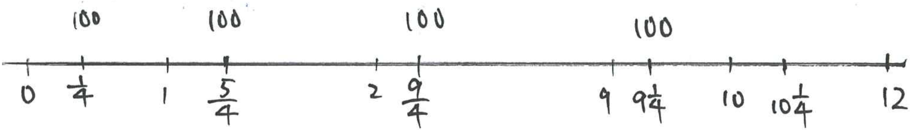
Solutions:

1. $$\ddot{s}_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + \ldots + (1+i)$$

2. $$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i)s_{\overline{n}|} = (1+i)\frac{(1+i)^n - 1}{i} \\ &= (1+i)(1+i)^n a_{\overline{n}|} \\ &= (1+i)^n\, \ddot{a}_{\overline{n}|} \end{aligned}$$

Example 11

1. Find the PV of a series of 10 payments of \$100 at yearly intervals. The first payment is due in 3 months' time. The interest rate is 8% per annum effective.
2. From first principles, find the accumulated value of the series above at time 12 years.
3. Check that your answers are consistent.



Solutions:

1. The PV is

$$
\begin{aligned}
& 100\left(v^{\frac{1}{4}} + v^{1\frac{1}{4}} + \ldots + v^{9\frac{1}{4}}\right) \\
= \; & 100v^{\frac{1}{4}}\left(1 + v + \ldots + v^{9}\right) \\
= \; & 100v^{\frac{1}{4}}\, \ddot{a}_{\overline{10}|} \\
= \; & 100(1.08^{-0.25})\frac{1 - 1.08^{-10}}{0.08}1.08 \\
= \; & \$710.88.
\end{aligned}
$$

2. The accumulated value is

$$
\begin{aligned}
& 100\left((1+i)^{11\frac{3}{4}}+(1+i)^{10\frac{3}{4}}+\ldots+(1+i)^{2\frac{3}{4}}\right) \\
= \; & 100(1+i)^{2\frac{3}{4}}\left((1+i)^{9}+(1+i)^{8}+\ldots+1\right) \\
= \; & 100(1+i)^{2\frac{3}{4}} \; s_{\overline{10|}} \\
= \; & 100(1.08)^{2\frac{3}{4}} \; \frac{1.08^{10}-1}{0.08} \\
= \; & \$1,790.11.
\end{aligned}
$$

3. The valuation dates are 12 years apart. We can verify that

$$710.88(1.08^{12}) = 1,790.11.$$

9 Further variations on annuities

- Annuities payable less frequently than once a year
- Annuities with variable interest rate
- Annuities with variable payment amounts

Annuities payable less frequently than once a year

- Consider an annuity with regular payments of \$200 payable at two yearly intervals in arrears.
- The final payment is made after 20 years.
- Find the PV at 6% per annum effective.

Method 1: Using First Principles

- Summing up the PVs of the individual payments, we get

$$
\begin{aligned}
\text{PV} &= 200\left(1.06^{-2} + 1.06^{-4} + \ldots + 1.06^{-20}\right) \\
&= 200 \times 1.06^{-2}\left(1 + 1.06^{-2} + 1.06^{-4} + \cdots + 1.06^{-18}\right) \\
&= 200 \times 1.06^{-2}\frac{(1 - 1.06^{-20})}{1 - 1.06^{-2}} = \$1{,}113.58.
\end{aligned}
$$

Method 2: Change the time unit

- New time unit is two-year
- The effective 2-year interest rate is j such that

$$(1+j) = (1+i)^2 = 1.06^2 \Longrightarrow j = 12.36\%.$$

- The 2-year discount factor is $v_i^2 = 1/(1+j) = v_j$.

Then the PV of this annuity is

$$200 \times a_{\overline{10|}\ @j} = 200\frac{1-v_i^{20}}{j} = 200\frac{1-v_j^{10}}{j} = \$1{,}113.58.$$

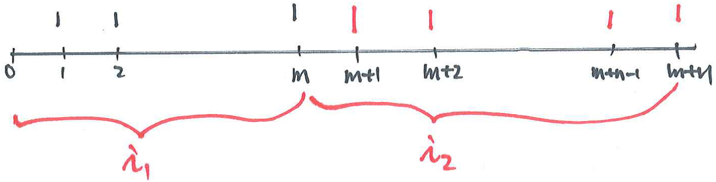
9 Further variations on annuities

- Annuities payable less frequently than once a year
- Annuities with variable interest rate
- Annuities with variable payment amounts

Annuities with variable interest rate

Consider the following $m+n$ year annuity in arrears:

- If the effective annual interest rate is i throughout the term, then the PV of the annuity is $a_{\overline{n+m}|\ i}$, and the accumulated value is $s_{\overline{n+m}|\ i}$.



- If the interest rate is i_1 by time m years, and the interest rate is i_2 after time m years, then the PV of the annuity is

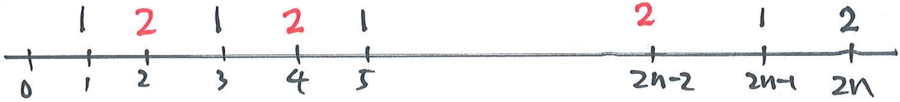
$$a_{\overline{m}|\ i_1} + v_{i_1}^m a_{\overline{n}|\ i_2}$$

9 Further variations on annuities

- Annuities payable less frequently than once a year
- Annuities with variable interest rate
- Annuities with variable payment amounts

Annuities with variable payment amounts

Consider the following annuity in arrears with term $2n$ years with constant annual effective interest rate i:



Define $j = (1+i)^2 - 1$. The PV of this annuity is

$$a_{\overline{2n}|\ i} + a_{\overline{n}|\ j} = (1+i)a_{\overline{n}|\ j} + 2a_{\overline{n}|\ j} = \ldots$$

References I

Atkinson, M. E., and David C. M. Dickson. 2011. *An Introduction to Actuarial Studies*. 2nd ed. Edward Elgar.